IND522
Advanced Statistical Modelling

Fall 2025

Professor Orhan FEYZİOĞLU

Assignment $\sharp$2: Due October 20

1) Solve the equations

$$\begin{aligned} 2x - y + z &= 4 \\ x + y + z &= 3 \\ 3x - y - z &= 1 \end{aligned}$$

using the MATLAB left division operator (/). Check your solution by computing the residual. Also compute the determinant (`det`) and the condition estimator (`rcond`). What do you conclude?

2) This problem, suggested by R.V. Andree, demonstrates ill conditioning (where small changes in the coefficients cause large changes in the solution). Use the MATLAB left division operator (/) to show that the solution of the system

$$\begin{aligned} x + 5.000y &= 17.0 \\ 1.5x + 7.501y &= 25.503 \end{aligned}$$

is $x = 2$, $y = 3$. Compute the residual.

Now change the term on the right-hand side of the second equation to 25.501, a change of about one part in 12000, and find the new solution and the residual. The solution is completely different. Also try changing this term to 25.502, 25.504, etc. If the coefficients are subject to experimental errors, the solution is clearly meaningless. Use `rcond` to find the condition estimator and `det` to compute the determinant. Do these values confirm ill conditioning?

Another way to anticipate ill conditioning is to perform a *sensitivity analysis* on the coefficients: change them all in turn by the same small percentage, and observe what effect this has on the solution.

3) If you are familiar with *Gauss reduction* it is an excellent programming exercise to code a Gauss reduction directly with operations on the rows of the augmented coefficient matrix. See if you can write a function

```
x = mygauss(a, b)
```

to solve the general system $\mathbf{Ax} = \mathbf{b}$. Skillful use of the colon operator in the row operations can reduce the code to a few lines! Test it on $\mathbf{A}$ and $\mathbf{b}$ with random entries, and on the systems in Questions 1 and 2.

4) Write your own function to compute the exponential function directly from the Taylor series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

The series should end when the last term is less than 10^{-6}. Test your function against the built-in function `exp`, but be careful not to make x too large—this could cause rounding error.

5) If a random variable X is distributed normally with zero mean and unit standard deviation, the probability that $0 \le X \le x$ is given by the standard normal function $\Phi(x)$. This is usually looked up in tables, but it may be approximated as follows:

$$\Phi(x) = 0.5 - r(at + bt^2 + ct^3),$$

where $a = 0.4361836$, $b = -0.1201676$, $c = 0.937298$, $r = \exp(-0.5x^2)/\sqrt{2\pi}$, and $t = 1/(1 + 0.3326x)$. Write a function to compute $\Phi(x)$, and use it in a program to write out its values for $0 \le x \le 4$ in steps of 0.1. Check: $\Phi(1) = 0.3413$.

6) There are many formulae for computing π (the ratio of a circle's circumference to its diameter). The simplest is

$$\frac{\pi}{4} = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots \tag{1}$$

which comes from putting $x = 1$ in the series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots \tag{2}$$

a. Write a program to compute π using Equation (1). Use as many terms in the series as your computer will reasonably allow (start modestly, with 100 terms, say, and re-run your program with more and more each time). You should find that the series converges very slowly, i.e. it takes a lot of terms to get fairly close to π.

b. Rearranging the series speeds up the convergence:

$$\frac{\pi}{8} = \frac{1}{1 \times 3} + \frac{1}{5 \times 7} + \frac{1}{9 \times 11} \ldots$$

Write a program to compute π using this series instead. You should find that you need fewer terms to reach the same level of accuracy that you got in (a).

c. One of the fastest series for π is

$$\frac{\pi}{4} = 6 \arctan \frac{1}{8} + 2 \arctan \frac{1}{57} + \arctan \frac{1}{239}.$$

Use this formula to compute π. Don't use the built-in function `atan` to compute the arctangents, since that would be cheating. Rather use Equation (2).

d. Can you vectorize any of your solutions (if you haven't already)?

7) The following method of computing π is due to Archimedes:

1. let $A = 1$ and $N = 6$

2. Repeat 10 times, say:

Replace N by $2N$
Replace A by $\left[2 - \sqrt{4 - A^2}\right]^{1/2}$
Let $L = NA/2$
Let $U = L/\sqrt{1 - A^2/2}$
Let $P = (U + L)/2$ (estimate of π)
Let $E = (U - L)/2$ (estimate of error)
Print N, P, E

3. Stop.

Write a program to implement the algorithm.

8) If an amount of money A is invested for k years at a nominal annual interest rate r (expressed as a decimal fraction), the value V of the investment after k years is given by

$$V = A\,(1 + r/n)^{nk}\,.$$

where n is the number of compounding periods per year. Write a program to compute V as n gets larger and larger, i.e. as the compounding periods become more and more frequent, like monthly, daily, hourly, etc. Take $A = 1000$, $r = 4$ per cent and $k = 10$ years. You should observe that your output gradually approaches a limit. *Hint*: use a for loop which doubles n each time, starting with $n = 1$.

Also compute the value of the formula Ae^{rk} for the same values of A, r and k (use the function `exp`), and compare this value with the values of V computed above. What do you conclude?